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Noether's theorem in generalized mechanics

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Abstract. Noether's theorem in the calculus of variations is extended to the case of a lagrangian density containing higher order derivatives. It is then demonstrated how some recently given results of generalized mechanics can be seen as consequences of conservation equations resulting from the extended version of Noether's theorem.

1. Introduction

Within the calculus of variations the well known theorem of Noether (see eg Gelfand and Fomin 1965), plays a fundamental role. It gives in a systematic manner a connection between the conservation laws of a physical theory and the invariances of the corresponding variational integral, whose Euler equations are the equations of the theory. However, the emphasis in the calculus of variations has been on problems, whose lagrangian density contains first derivatives at most of the field variable. In view of the recent interest in variational problems involving higher derivatives (eg Borneas 1969, 1972, Coelho de Souza and Rodrigues 1969) it is important to formulate the very powerful theorem of Noether so as to apply it to these extended problems. This is all the more so since there has also been renewed interest in Noether's theorem in various contexts (cf Tavel 1971, Lévy-Leblond 1971).

A generalized form of the theorem will be derived, which is shown to reduce to the conventional Noether's theorem, when higher order derivatives are absent.

The extended theorem is then applied to the theory of generalized mechanics put forward by Borneas (1969, 1972). It is shown that the generalized expressions for energy and momentum given by Borneas can be derived from conservation laws following from the modified form of Noether's theorem. Finally the relation between the higher momenta appearing in the theory of generalized mechanics and the invariance properties of the variational integral is investigated.

2. The generalized theorem of Noether

In this section we will derive the appropriate generalization of Noether's theorem. Notations and method of approach will closely follow those used by Gelfand and Fomin (1965).

Consider the following functional $J[y]$ given by

$$J[y] = \int_{x_0}^{x_1} L(x, y(x), \dots \mathrm{D}^n y(x))\, \mathrm{d}x \tag{1}$$

where $D^i y(x) = d^i y(x)/dx^i$, $i = 1, 2, \ldots n$, and L is the lagrangian density corresponding to a certain physical variational problem. We now subject the functional $J[y]$ to the following variation:

$$x \to x^* = x + \delta x$$
$$y(x) \to y^*(x^*) = y(x) + \delta y(x). \tag{2}$$

For convenience we also introduce the quantity $\bar{\delta} y(x)$ given by

$$\bar{\delta} y(x) = y^*(x) - y(x) \simeq \delta y - \delta x \mathrm{D} y. \tag{3}$$

In contrast to the ordinary variation δy, the barred variation $\bar{\delta} y$ obeys the commutation relation

$$\mathrm{D}^j(\bar{\delta} y) = \bar{\delta}(\mathrm{D}^j y). \tag{4}$$

The variation of J, expressed in terms of $\bar{\delta} y$ rather than δy, we denote $\bar{\delta} J$ and is given by (see Appendix)

$$\bar{\delta} J = \int_{x_0}^{x_1} \frac{\delta L}{\delta y} \bar{\delta} y \, \mathrm{d}x + \left[L\, \delta x + \sum_{j=1}^{n} P_j \, \bar{\delta}(\mathrm{D}^{j-1} y) \right]_{x_0}^{x_1} \tag{5}$$

where

$$\frac{\delta L}{\delta y} = \sum_{i=0}^{n} (-\mathrm{D})^i \frac{\partial L}{\partial(\mathrm{D}^i y)}$$
$$P_j = \sum_{i=0}^{n-j} (-\mathrm{D})^i \frac{\partial L}{\partial(\mathrm{D}^{i+j} y)}. \tag{6}$$

Assuming for a moment fixed boundary conditions, we obtain the Euler equation

$$\frac{\delta L}{\delta y} = 0. \tag{7}$$

In the following analysis we shall be working within the set of all functions satisfying equation (7). This implies that

$$\bar{\delta} J = \left[L\, \delta x + \sum_{j=1}^{n} P_j \, \bar{\delta}(\mathrm{D}^{j-1} y) \right]_{x_0}^{x_1}. \tag{8}$$

With the use of equations (3) and (4) we obtain the first variation δJ as

$$\delta J = \left[L\, \delta x + \sum_{j=1}^{n} P_j \mathrm{D}^{j-1} (\delta y - \delta x \mathrm{D} y) \right]_{x_0}^{x_1}. \tag{9}$$

Changing our point of view, we regard equation (2) as the result of an infinitesimal transformation rather than a variation. Write instead

$$\left.\begin{aligned} x \to x^* &= \Phi(x, y, \mathrm{D}y, \ldots \mathrm{D}^n y, \epsilon) = x + \epsilon \phi(x, y, \mathrm{D}y, \ldots \mathrm{D}^n y) + \mathrm{O}(\epsilon) \\ y \to y^* &= \Psi(x, y, \mathrm{D}y, \ldots \mathrm{D}^n y, \epsilon) = y + \epsilon \psi(x, y, \mathrm{D}y, \ldots \mathrm{D}^n y) + \mathrm{O}(\epsilon) \end{aligned}\right. \tag{10}$$

where ϵ is an infinitesimal constant parameter. Comparing equations (2) and (10) we get

$$\delta x = \epsilon \phi \qquad \delta y = \epsilon \psi. \tag{11}$$

We now pick out for special study those infinitesimal transformations, as given by equation (10), which leave the functional $J[y]$ invariant, that is, we have

$$J[y] = J[y^*] = \int_{x_0^*}^{x_1^*} L(x^*, \mathrm{D}y^*(x^*), \ldots \mathrm{D}^n y^*(x^*)) \, \mathrm{d}x^*. \tag{12}$$

In this case the variation δJ must vanish and we get the conservation equation

$$L\phi + \sum_{j=1}^{n} P_j \mathrm{D}^{j-1}(\psi - \phi \mathrm{D}y) = \text{constant}. \tag{13}$$

It is easily realized from equation (6), that if L does not depend on derivatives of $y(x)$ of higher order than first we have

$$P_j = \begin{cases} 0 & j = 2, 3, \ldots, n \\ \dfrac{\partial L}{\partial(\mathrm{D}y)} & j = 1 \end{cases} \tag{14}$$

and equation (13) reduces to

$$\psi \frac{\partial L}{\partial(\mathrm{D}y)} + \phi \left(L - \mathrm{D}y \frac{\partial L}{\partial(\mathrm{D}y)} \right) = \text{constant} \tag{15}$$

which is the conventional form of Noether's theorem.

Thus equation (13) constitutes a generalized form of Noether's theorem, which includes terms due to the presence of higher order derivatives in the lagrangian. The extension to the case when L depends on several functions $y_k(x), k = 1, 2, \ldots, N$, is straightforward and we give it without a formal proof. Write

$$L = L(x, y(x), \mathrm{D}y(x), \ldots, \mathrm{D}^n y(x)) \tag{16}$$

where $y(x) = (y_1(x), y_2(x), \ldots, y_N(x))$. The transformation corresponding to equation (10) is

$$\begin{aligned} x \to x^* &= x + \epsilon\phi(x, y(x), \ldots, \mathrm{D}^n y(x)) + \mathrm{O}(\epsilon) \\ y_k \to y_k^*(x^*) &= y_k(x) + \epsilon\psi_k(x, y(x), \ldots, \mathrm{D}^n y(x)) + \mathrm{O}(\epsilon). \end{aligned} \tag{17}$$

The quantities P_j are generalized to

$$P_j^{(k)} = \sum_{i=0}^{n-j} (-\mathrm{D})^i \frac{\partial L}{\partial(\mathrm{D}^{i+j} y_k)} \qquad \begin{aligned} j &= 1, 2, \ldots, n \\ k &= 1, 2, \ldots, N. \end{aligned} \tag{18}$$

Finally equation (13) will now read

$$L\phi + \sum_{k=1}^{N} \sum_{j=1}^{n} P_j^{(k)} \mathrm{D}^{j-1}(\psi_k - \phi \mathrm{D}y_k) = \text{constant}. \tag{19}$$

For the sake of simplicity we have assumed that L contains derivatives of the same maximum order of all appearing functions. The further extension to the case when L depends on derivatives of y_k up to order n_k, $k = 1, 2, \ldots, N$, is equally straightforward, but will not be given here.

3. Applications and comparisons

3.1. Generalized hamiltonian and momenta

As an application of the generalized Noether's theorem derived in the preceding section, we shall study two simple transformations leading to important conservation equations and compare our results with the results given by Borneas (1972).

(i) Consider the following transformation corresponding to a translation of the independent variable, namely,

$$x^* = x + \epsilon \qquad y_k^* = y_k \qquad k = 1, 2, \ldots, N. \tag{20}$$

We get, (cf equation (11))

$$\phi = 1 \qquad \psi_k = 0 \qquad k = 1, 2, \ldots, N \tag{21}$$

which, inserted into equation (19), gives

$$L - \sum_{k=1}^{N} \sum_{j=1}^{n} P_j^{(k)} \mathrm{D}^j y_k = \text{constant}. \tag{22}$$

This is essentially the generalized hamiltonian $\tilde{H}$ given by Borneas (1972). Actually we have

$$\tilde{H} = \sum_{k=1}^{N} \sum_{j=1}^{n} P_j^{(k)} \mathrm{D}^j y_k - L. \tag{23}$$

The functional J is seen to be invariant under the transformation equation (20), if we assume that the lagrangian does not depend explicitly on x. In the simple case expressed by equation (15) a translation analogous to equation (20) leads to the conservation of the hamiltonian expression

$$H = \mathrm{D}y \frac{\partial L}{\partial(\mathrm{D}y)} - L. \tag{24}$$

Furthermore $\tilde{H}$ reduces to H in the absence of higher derivatives in L. Thus, there are ample reasons for calling H the generalized hamiltonian.

(ii) Next we study a translation of the dependent variable $y_k(x)$ as follows:

$$x^* = x \qquad y_k^* = y_k + \epsilon\, \delta_{kl} \qquad k = 1, 2, \ldots, N \tag{25}$$

where δ_{kl} is the usual Kronecker delta and l is an arbitrary number between 1 and N. This implies

$$\phi = 0 \qquad \psi_k = \delta_{kl} \qquad k = 1, 2, \ldots, N. \tag{26}$$

If we insert this into equation (19) we get

$$P_1^{(l)} = \sum_{i=0}^{n-1} (-\mathrm{D}^i) \frac{\partial L}{\partial(\mathrm{D}^{i+1} y_l)} = \text{constant}. \tag{27}$$

However, l is arbitrary and we obtain the result (cf Borneas 1972)

$$P_1^{(k)} = \text{constant} \qquad k = 1, 2, \ldots, N. \tag{28}$$

With arguments similar to those leading to the identification of $\tilde{H}$ with H, we can justify the names generalized momenta for the $P_j^{(k)}$.

3.2. *Higher momenta*

Although all first momenta $P_1^{(k)}$, $k = 1, 2, \ldots N$ are conserved as a consequence of the invariance of J under the transformation equation (25), this has no consequences for the higher momenta $P_j^{(k)}$, $j = 2, 3, \ldots, n$, $k = 1, 2, \ldots, N$. A natural question (not discussed by Borneas) is: Under what conditions are the higher momenta conserved? And from our point of view: What is the relation to the invariance properties of the functional?

In this section we will discuss these questions and also compare them with an algebraic approach based on a set of identities which constitute a recurrence formula for the determination of P_j (we shall take $N = 1$ for convenience).

The functional J is easily seen to be invariant under the transformation equation (25) if L does not depend explicitly on y. As a generalization of this, we assume that L does not depend explicitly on $D^j y$, $j = 0, 1, 2, \ldots, m < n$, which implies that

$$\frac{\partial L}{\partial(D^j y)} = 0 \qquad \text{for } j = 0, 1, \ldots, m. \tag{29}$$

First we shall use the following identities (cf Hadamard 1910):

$$\begin{aligned} \frac{dP_{j+1}}{dx} &= \frac{\partial L}{\partial(D^j y)} - P_j \qquad j = 1, \ldots, n-1 \\ \frac{dP_1}{dx} &= \frac{\partial L}{\partial y} - \frac{\delta L}{\delta y}. \end{aligned} \tag{30}$$

When L satisfies the conditions expressed by equation (29), we can integrate equation (30) recursively and obtain

$$P_{j+1} = \sum_{k=1}^{j+1} \frac{(-1)^{k+1} c_k}{k!} x^{j+1-k} \qquad j = 0, 1, \ldots, m \tag{31}$$

where c_k are constants. From this result we draw the conclusion that P_{j+1} is constant if and only if $c_k = 0$ for $k = 1, 2, \ldots, j$, which is equivalent to the condition that all momenta of order less than $j+1$ shall vanish.

Comparing this with the transformation approach we see that, provided equation (29) is fulfilled, J must be invariant under the following set of transformations:

$$x \to x^* = x \qquad y \to y^* = y + \epsilon x^j \qquad j = 0, 1, 2, \ldots, m. \tag{32}$$

This yields $\phi = 0$ and $\psi = x^j$, which, inserted into Noether's generalized theorem (equation (13)), gives

$$\sum_{k=1}^{n} P_k D^{k-1} \psi = \text{constant}. \tag{33}$$

But $D^{k-1}\psi = 0$ for $k - 1 \geqslant j + 1$ and equation (33) reduces to

$$\sum_{k=1}^{j+1} P_k b_k x^{j+1-k} = \text{constant} \tag{34}$$

where b_k are constants. Equation (34) can be written

$$P_{j+1} = -\sum_{k=1}^{j} P_k b_k x^{j+1-k} + \text{constant}. \tag{35}$$

This implies that P_{j+1} is constant if and only if all P_k vanish for $k \leqslant j$ and that P_{j+1} generally is a polynomial in x of degree j, a result which coincides with that obtained by algebraic means.

4. Conclusions

The important theorem of Noether has been extended to take into account higher order derivatives in the lagrangian. When applied to the area of generalized mechanics it makes possible a systematic study of conservation equations, which as special cases yields the conservation of energy and first momenta previously given by Borneas (1972). Finally the generalized theorem is shown to be a convenient tool when discussing the problem of the conservation of higher momenta.

Appendix

We derive the appropriate expression for the first (barred) variation of the functional $J[y]$ (equation (1)) assuming free boundaries. The derivation demonstrates the very convenient use that can be made of the concepts of bilinear concomitant and adjoint operator. The variation of the functional $J[y]$ (equation (1)) is

$$\bar{\delta} J = \int_{x_0}^{x_1} \bar{\delta} L \,\mathrm{d}x + \int_{x_0}^{x_1} L\, \delta\,(\mathrm{d}x). \tag{A.1}$$

The second integral of equation (A.1) can be evaluated at once to give

$$\int_{x_0}^{x_1} L\, \delta\,(\mathrm{d}x) = [L\, \delta x]_{x_0}^{x_1}. \tag{A2}$$

The integrand of the first integral in equation (A.1) is obtained by means of a Taylor expansion

$$\bar{\delta} L = \sum_{k=0}^{n} \frac{\partial L}{\partial(\mathrm{D}^k y)} \bar{\delta}(\mathrm{D}^k y). \tag{A.3}$$

Consider the following special formulation of the Lagrange identity (cf Ince 1944):

$$v\mathrm{D}^k u = \mathrm{D}[Q_k(u, v)] + u\overline{\mathrm{D}^k} v \tag{A.4}$$

where u and v are arbitrary smooth functions, $Q_k(u, v)$ is the bilinear concomitant of, and $\overline{\mathrm{D}^k}$ the operator algebraically adjoint to, D^k. These are given by

$$Q_{k+1}(u, v) = \sum_{l=0}^{k} (-1)^{k-l} \mathrm{D}^l u \mathrm{D}^{k-l} v \qquad \overline{\mathrm{D}^k} = (-1)^k \mathrm{D}^k. \tag{A.5}$$

With the use of the commutation relations (equation (4)) and equation (A.5) we obtain

$$\frac{\partial L}{\partial(\mathrm{D}^k y)} \bar{\delta}(\mathrm{D}^k y) = \frac{\partial L}{\partial(\mathrm{D}^k y)} \mathrm{D}^k(\bar{\delta} y) = \bar{\delta} y(-\mathrm{D})^k \left[\frac{\partial L}{\partial(\mathrm{D}^k y)}\right] + \mathrm{D}\left[Q_k\left(\bar{\delta} y, \frac{\partial L}{\partial(\mathrm{D}^k y)}\right)\right]. \tag{A.6}$$

We introduce the notation

$$Q(y, L) = \sum_{k=1}^{n} Q_k\left(\bar{\delta} y, \frac{\partial L}{\partial(\mathrm{D}^k y)}\right). \tag{A.7}$$

Using the definition of the variational derivative defined by equation (6) we obtain for the barred variation of L:

$$\bar{\delta} L = \frac{\delta L}{\delta y} \bar{\delta} y + \mathrm{D}[Q(y, L)]. \tag{A.8}$$

$Q(y, L)$ can be put into a more convenient form as follows. Write first

$$Q(y, L) = \sum_{k=1}^{n} Q_k\left(\bar{\delta} y, \frac{\partial L}{\partial(\mathrm{D}^k y)}\right) = \sum_{k=1}^{n} \sum_{l=0}^{k-1} (-1)^{k-l-1} \mathrm{D}^l(\bar{\delta} y) \mathrm{D}^{k-l-1}\left[\frac{\partial L}{\partial(\mathrm{D}^k y)}\right]. \tag{A.9}$$

Changing summation variables according to:

$k-l-1 = i$ and $l+1 = j$, we obtain

$$Q(y, L) = \sum_{j=1}^{n} \sum_{i=0}^{n-j} (-\mathrm{D})^i\left[\frac{\partial L}{\partial(\mathrm{D}^{i+j} y)}\right] \mathrm{D}^{j-1}(\bar{\delta} y). \tag{A.10}$$

We introduce the quantities P_j defined by equation (6). This yields

$$Q(y, L) = \sum_{j=1}^{n} P_j \mathrm{D}^{j-1}(\bar{\delta} y) \tag{A.11}$$

and summing up equations (A.1), (A.2), (A.8) and (A.11) we obtain

$$\bar{\delta} J = \int_{x_0}^{x_1} \frac{\delta L}{\delta y} \bar{\delta} y \, \mathrm{d}x + \left[L \, \delta x + \sum_{j=1}^{n} P_j \bar{\delta}(\mathrm{D}^{j-1} y)\right]_{x_0}^{x_1} \tag{A.12}$$

which is the desired result (equation (5)).

References

Borneas M 1969 *Phys. Rev.* **186** 1299–303

——1972 *Am. J. Phys.* **40** 248–51

Coelho de Souza L M C and Rodrigues P R 1969 *J. Phys.* A: *Gen. Phys.* **2** 304–10

Gelfand I M and Fomin S V 1965 *Calculus of Variations* (New York: Prentice-Hall)

Hadamard J 1910 *Lecons sur le Calcul des Variations* (Paris: Hermann et Fils)

Ince E L 1944 *Ordinary Differential Equations* (New York: Dover)

Lévy-Leblond J-M 1971 *Am. J. Phys.* **39** 502–6

Tavel M 1971 *Transp. Theor. Stat. Phys.* **1** 271–85